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Optimal Dynamic Investment with Learning by Doing in the Adjustment Cost Function[1]

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Abstract: In an optimal control model of a firm's dynamic investment policy, we analyze the effects of including learning by doing in the adjustment cost function (the larger the existing capital stock, the smaller the cost of installing an additional unit of capital stock).

Key Words: Optimal Control, Dynamic Investment Policy, Adjustment Costs, Learning by Doing.

1 Introduction

The concept of adjustment costs in investment theory appears in several studies of a firm's optimal investment behaviour [e.g., Gould (1968)]. The assumption is that capital inputs are adjustable, but at an expense, the adjustment cost. One motivation for the occurrence of such a cost is temporary decreases of productivity due to reorganization of the production line upon the installation of new machinery. The basic hypothesis of this paper is that there may be a learning-by-doing effect in the installation process.

Let $K = K(t)$ denote the firm's stock of capital at time t, $I = I(t)$ its gross investment rate, and C the adjustment cost function. The specification $C = C(I)$ has often been employed but sometimes it has been argued that the adjustment cost function should rather depend on the relative rate of accumulation I/K [Hayashi (1982)]. Lucas (1967) suggested the specification $C = C(K, I)$ but without any particular motivation.

This paper proposes a specific motivation for the choice of an adjustment cost function, having both the investment rate and the capital stock as arguments, i.e., $C = C(K, I)$. Section 2 presents the dynamic optimization problem and Section 3 contains the analysis and the results.

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2 The Optimal Control Model

The differential equation for net investment dK/dt is

$$dK/dt = \dot{K} = I - aK; \qquad K(0) = K_0 > 0 \text{ and given} \tag{1}$$

where a is the nonnegative depreciation rate. Integration in (1) shows that the stock of capital

$$K(t) = \exp\{-at\}K_0 + \int_0^t \exp\{-a(t-v)\}I(v)\,dv$$

can be interpreted to represent cumulative investment, corrected for depreciation.

Specifying the adjustment cost as $C = C(K, I) = c(K)I$, where $c(K)$ represents the *unit* adjustment cost, the learning hypothesis is

$$c'(K) < 0 \tag{2}$$

which states that *the larger the current capital stock, the larger the installation experience and hence the smaller the cost of installing an additional unit of capital stock.* Depreciation can be seen as providing a "forgetfulness" effect such that recent investments contribute more to the capital stock, and thus installation experience, than older ones.

We assume $c(K)$ twice differentiable and

$$\begin{aligned} &c(K) > 0 \qquad \text{and} \qquad c''(K) > 0 \quad \text{for all } K \geq 0 \\ &c(K) \to c_L = \text{const.} > 0 \quad \text{for } K \to \infty\ . \qquad c_L < c(0)\ . \end{aligned} \tag{3}$$

Thus, installation experience is subject to diminishing returns but there is a limit to the decrease in $c(K)$.

Gross earnings of the firm are given by an instantaneous revenue function $S(K)$ [defined as revenue after maximization with respect to variable inputs] which is C^2 and satisfies $S(K) > 0$ for $K > 0$, $S(0) = 0$, $S'(K) > 0$ for $K \geq 0$, $S'(K) \to 0$ for $K \to \infty$, and $S''(K) < 0$.

The objective functional of the firm can now be stated as

$$J(I, K) = \int_0^\infty \exp\{-it\}[S(K) - I - c(K)I]\,dt \tag{4}$$

where $i > 0$ is the firm's time preference rate. The problem is to determine the control variable $I(t)$, $t \in [0, \infty)$, so as to maximize $J(I, K)$ subject to (1) as well as

$$I \geq 0 \qquad \text{for all } t \in [0, \infty) \tag{5}$$

$$S(K) - I - c(K)I \geq 0 \qquad \text{for all admissible } (I, K) \ . \tag{6}$$

Constraint (5) requires irreversibility of investment whereas (6), being somewhat more restrictive, means that the firm cannot borrow additional money to finance its operations. Constraint (6) also excludes the possibility of issuing new shares. Thus, the only source of additional finance is retained earnings.

3 Analysis of the Control Problem

Define $S(K)/[1 + c(K)] := I_m(K)$. The control region is given by $I \in [0, I_m(K)]$. Substituting from (1) into (4) yields

$$J(K) = \int_0^\infty \exp\{-it\}[S(K) - aK(1 + c(K)) - (1 + c(K))\dot{K}]\, dt \ .$$

The maximization of $J(K)$ with respect to K, subject to $K \in [0, \infty)$, is a variational problem to which we add the initial condition $K(0) = K_0$, and the feasibility constraint

$$\dot{K} \in \Omega(K) \ ; \qquad \Omega(K) = \{I - aK | I \in [0, I_m(K)]\} = [-aK, I_m(K) - aK] \ .$$

Let K^* be the singular level of K. Due to the time-invariance of the problem, the level K^* is constant. It is a solution to

$$S'(K) - c'(K)aK = (i + a)(1 + c(K)) \ . \tag{7}$$

To maintain $K(t)$ at the level K^*, the firm must apply the policy $I^* = aK^*$. The economic contents of (7) are as follows. Along the singular path (which will be the steady state solution), marginal revenue (LHS of (7)) equals marginal cost (RHS of (7)). Note that marginal revenue consists of the standard term $S'(K)$ plus the cost saving, $-c'(K)aK$, which arises due to learning. Without the cost saving term, (7) is standard (cf. Takayama (1985, pp. 698–699)). In what follows, define $G(K) := c'(K)aK + (i + a)(1 + c(K))$.

In (7), $S'(K)$ decreases as K increases, but $S'(K)$ remains positive for all K. Now, $S'(0) > G(0)$ if we *assume* that

$$S'(0)/(i + a) > 1 + c(0) \tag{8}$$

which is a standard condition [Takayama (1986, p. 702), Feichtinger Hartl (1986, p. 318)]. It states, roughly speaking, that the marginal revenue provided by the first unit of capital exceeds the marginal cost of that unit.

The singular level K^* is such that $S'(K^*) = G(K^*)$. Now, the derivative of $G(K)$ is $G'(K) = c''(K)aK + (i + 2a)c'(K)$ and we see that the existence of K^* is guaranteed since S' decreases monotonically and G must eventually increase since (by assumption) $c' \to 0$ for $K \to \infty$ whereas c'' is positive for all K.

Nonuniqueness of K^* may occur. To exclude this, a convenient assumption would be strict convexity of function $G(K)$, but notice that $G''(K)$ contains a third order derivative of the cost $c(K)$ and it seems difficult to provide an economically meaningful interpretation of the condition $G''(K) > 0$.

Consider Figure 1. Recall that $S'(0) > G(0)$ and that $G'(K)$ is increasing for sufficiently large values of K. Also note that $G'(0) < 0$ is negative. Now, if there are multiple equilibria, they must occur as sketched in Figure 1 (where there are three equilibria). Denote, in general, the equilibria by $K(1)^* < K(1)' < K(2)^* < K(2)' < \cdots < K(n)^*$ and observe that

$$S'(K_i) - G(K_i)\begin{Bmatrix} > \\ < \end{Bmatrix} 0 \qquad \text{if } K \in \begin{cases} (K(i-1)', K(i)^*) \text{ or } K < K(1)^* \\ (K(i)^*, K(i)') \text{ or } K > K(n)^* \end{cases} \tag{9}$$

for $i = 2, 3, \ldots, n - 1$. The function $S'(K) - G(K)$ is sketched in Figure 1.

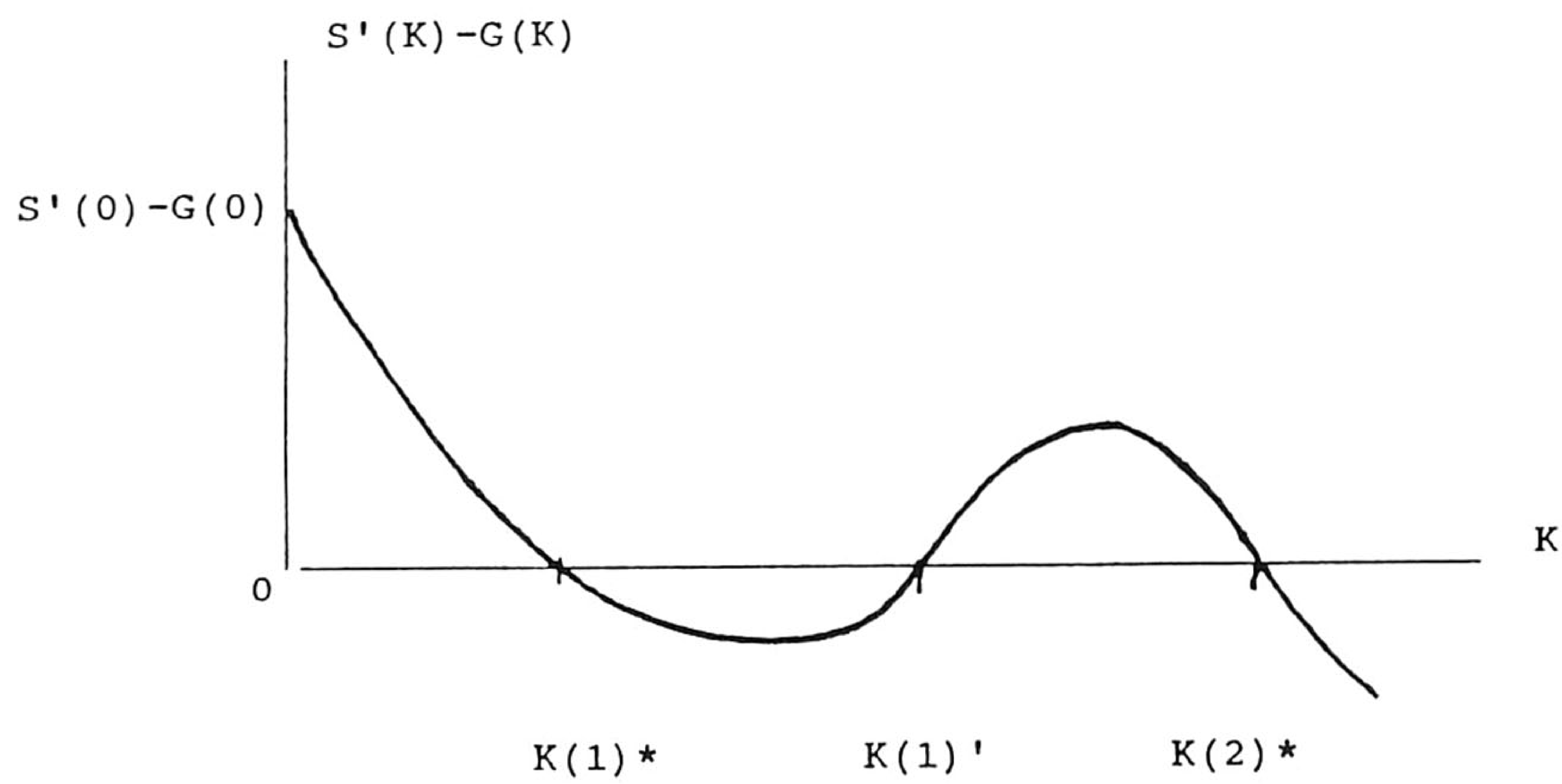

Fig. 1. A case of multiple equilibria

It is known [Feichtinger and Hartl (1986)] that the optimal trajectory $K(t)$ is an *MRAP* (Most Rapid Approach Path) from the initial state K_0 to one of the equilibria. Obviously, it depends on the particular location of K_0, which equilibrium it is optimal to approach.

Remark 1: The equilibrium (equilibria) must be feasible. Since any equilibrium, K^*, is time-independent, feasibility is assured if $0 \in \Omega(K^*)$. Thus, K^* is feasible if we *assume* that $I_m(K^*) - aK^* > 0 \leftrightarrow S(K^*) - aK^* - c(K^*)aK^* > 0$. This means that $I^* = aK^*$ is a value of I for which the constraint (6) does not bind. The assumption is often met in optimal control problems. Its implication is that the pair (K^*, I^*) provides an 'interior' solution, i.e., (6) holds with strict inequality. [If K^* were so large that the corresponding I^* violated (6), the turnpike K^* could not be followed]. With an infinite horizon, the following condition must also be satisfied for any feasible $K(t)$ [Feichtinger and Hartl (1986)].

$$\lim_{t \to \infty} \exp\{-it\} \int_{K(t)}^{K^*} -[1 + c(\sigma)]d\sigma \geq 0 \ .$$

To satisfy the condition, observe (i) the integrand is negative and bounded, (ii) K^* is finite (constant) and (iii) any feasible $K(t)$ is bounded (by zero) from below.

The optimal investment policy associated with an MRAP becomes

$$I^*(t) = \begin{cases} I_m(K) & \text{if } 0 < K(t) < K^* \\ aK^* & \text{if } K(t) = K^* \\ 0 & \text{if } K^* < K(t) \ . \end{cases} \tag{10}$$

In the plausible case $K_0 < K^*$ (i.e., the firm starts out with a relatively small amount of capital goods) it is optimal to invest at the maximal rate $I_m(K)$ on an initial interval, $[0, t^*]$ say. After having reached (at time t^*) the singular stock level K^*, investment is continued at the constant rate aK^*.

It is easily proved that $I_m(K)$ increases with K. This means that the growth process is self-increasing: as the stock of capital goods grows, this larger stock permits larger rates of maximal investment effort which, in turn, raises the stock even more. Thus, the path from K_0 to K^* is not only a *most rapid approach* path, but also a path of *accelerating speed of approach.*

To compare with former results, rewrite (7) as

$$S'(K^*) = i + a + (i + a)c(K^*) + c'(K^*)aK^* \ . \tag{11}$$

In the case of *zero* adjustment costs, the singular stock of capital goods is a solution, K_1 say, to the famous marginal productivity rule

$$S'(K_1) = i + a \tag{12}$$

where the right-hand side is the rent for capital; recall from (4) that the price of capital was put equal to one.

In the 'standard case' the adjustment cost function depends *only* on the *investment* rate. Thus, the cost equals $C(I)$ and the singular capital stock is given by

$$S'(K_s) = (i + a) + C'(aK_s)(i + a) \tag{13}$$

which is also a well known result [e.g., Nickell (1978, p. 30))].

To compare (13) and (11) we need to do the following. The adjustment cost function *with learning* is $C(K, I) = c(K)I$, being linear in I. Let the adjustment cost *without learning*, $C(I)$, also be linear, that is, $C(I) = mI$, where m is the constant marginal cost of adjustment such that $m = c(0)$ (since there is no learning). Then (13) becomes

$$S'(K_s) = (i + a) + (i + a)c(0) \ . \tag{13a}$$

Remark 2: (12) and (13) show, rather trivially, that due to the positive adjustment cost, the steady state K_s is smaller than K_1.

Comparing (11) and (12) shows that

$$K^* \lesseqqgtr K_1 \qquad \text{if } (i + a)/a \gtreqqless -c'(K^*)K^*/c(K^*) \tag{14}$$

where $-c'(K)K/c(K)$ represents the elasticity of the unit adjustment cost function. Thus, if the unit adjustment cost function $c(K)$ is 'elastic' at the point corresponding to $K = K^*$ (more precisely, having elasticity greater than $1 + i/a$), K^* will be greater than K_1. On the other hand, if $c(K)$ is 'unelastic' at $K = K^*$ (elasticity less than $1 + i/a$), K^* will be less than K_1. This makes sense because an 'elastic' unit adjustment cost function means that adjustment costs are more sensitive to learning; an increase in the capital stock results in a (relatively) large decrease of the unit adjustment cost function. Consequently, the (optimal) steady state value K^* will be (relatively) large.

Finally, comparing (11) and (13) shows that the equilibrium capital stock with learning exceeds the one without learning. This follows from the specificaton $m = c(0)$.

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